Lecture 11

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VOTING

Model: Let N of n voters, A of m alternatives / candidates.
Each voter $i \in N$ has a linear order $\succ_i$ over the alternatives (complete, transitive, antisymmetric) (aka a preference order or a ranking)
This is sometimes called "ordinal" preferences, vs. "cardinal preferences"

$a \succ_i b \Rightarrow$ i prefers a over b

rank$_i(a)$ $= 1 + |\{b : b \succ_i a\}|$

$L(A)$: set of all linear orders over A.

Input: $P = (\succ_1, \dots, \succ_n) \in L(A)^n$

Social Choice Fn : $F : P \to A$, chooses a single candidate

Social Choice Correspondence: $F : P \to 2^A$

Social Welfare Fn : $F : P \to L(A)$

Given P, $a, b \in A$:

$N(a,b)$ $= |\{i : a \succ_i b\}|$, then $N(a,b) + N(b,a) = n$

$\mu(a,b)$ $= N(a,b) - N(b,a)$
("margin" of a over b)

Voting Rules:

Positional scoring: vector $s = (s_1, \dots, s_m)$, $s_1 \geq s_2 \geq \dots \geq s_m$, not all equal, so that for $a \in A$, score$(a) = \sum_{i \in N} s_{\text{rank}_i(a)}$

Examples of voting rules that are positional scoring rules:

① Plurality : $s = (1, 0, \dots, 0)$
(elects the candidates top-ranked by largest # of voters)
Plurality(a) $= |\{i : \text{rank}_i(a) = 1\}|$

② Veto : $(1, 0, \dots, 0, -1)$

③ Borda : $(m-1, m-2, \dots, 1, 0)$

④ k-approval : $(\underbrace{1, 1, \dots, 1}_{k}, 0, 0, \dots, 0)$

Some other voting rules:

① Plurality w/ run off.
Top 2 candidates w/ highest Plurality have a pairwise majority of voters

Eg : 3x $a \succ b \succ c \succ d$
3x $b \succ a \succ c \succ d$
$c \succ b \succ a \succ d$
2x $d \succ c \succ b \succ a$
b wins in round 2.

② Single transferable vote:
Iteratively remove candidates w/ lowest Plurality score

③ Kemeny Ranking:
For 2 linear orders $\succ, \succ'$, define the Kendall-Tau distance:
$KT(\succ, \succ') = \frac{1}{2} |\{(a,b) : a \succ b \text{ & } b \succ' a\}|$

The Kemeny ranking returned $\succ^*$ minimizes $\sum_i KT(\succ^*, \succ_i)$

Condorcet Paradox:

Say a beats b if $\mu(a,b) > 0$.
A Condorcet winner is a candidate a that beats b $\forall b \neq a$.
(thus $\forall b \neq a$, a majority of voters prefer a to b)

Condorcet's Paradox : three ways not to be a Condorcet winner.

Example:
$a \succ b \succ c$
$b \succ c \succ a$
$c \succ a \succ b$

Now c beats a, a beats b, b beats c
(so social choice correspondence)

A voting rule F is Condorcet consistent, if whenever P has a Condorcet winner, $F(P)$ = the Condorcet winner.

Axioms:

1. Anonymity : $F(P)$ is unchanged if voters are permuted.
2. Neutrality : Let $\pi : A \to A$ be a bijection. Then F is neutral if $F(\pi(P)) = \pi(F(P))$
3. Pareto efficiency : If $b \succ_i a$ $\forall i$, then $a \notin F(P)$
4. Condorcet consistency
5. Consistency : Let P_1, P_2 be profiles over disjoint sets of voters. Then if $F(P_1) \cap F(P_2) \neq \emptyset$, then $F(P_1, P_2) = F(P_1) \cap F(P_2)$
6. Monotonicity : Given P, $a \in A$, $i \in N$, let P' be obtained by moving a up in i's ranking, keeping the relative order of other candidates unchanged. Then $a \in F(P) \Rightarrow a \in F(P')$.

Example 1: Borda is not Condorcet-consistent

2x $a \succ c \succ d \succ b$	score(a) = $6 + 6 = 12$
3x $b \succ a \succ c \succ d$	score(b) = 9

Example 2: STV is not Monotone

P1: 3x $a \succ c \succ b$
3x $b \succ c \succ a$
2x $c \succ a \succ b$
a

P2: 3x $a \succ c \succ b$
2x $c \succ a \succ b$
$b \succ a \succ c$
a

c

Now of the rules we've studied so far, all the properties listed.

Theorem: No positional scoring rule is Condorcet-consistent for $m \geq 3$

Example: 3x $a \succ b \succ c$
2x $b \succ c \succ a$
$c \succ a \succ b$
$b \succ a \succ c$

a is Condorcet winner.

① $s_2 > s_3$: a: $3s_1 + 2s_2 + 2s_3$
b: $3s_1 + 3s_2 + s_3$, so b beats a

② $s_1 > s_2 = s_3$: a & b tie

7. Continuity : F is continuous if : let P_1, P_2 be profiles over 2 different sets of voters, & $F(P_1) = \{a\}$. Then $\exists k \in \mathbb{Z}_+$ s.t. $F(P_1 + kP_2) = \{a\}$

Theorem : A SCC F is anonymous, neutral, consistent & continuous iff F is a positional scoring rule.
(w/o proof)

Corollary: No SCC F satisfies A, N, C, continuity & Condorcet-consistency.